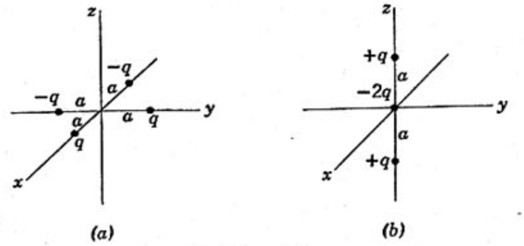
HOMEWORK #1

PROBLEMS

4.1 Calculate the multipole moments of the charge distributions shown below. in each case find the at the very least, the moments through quadrupole.



Problem 4.1

(c) For the charge distribution (b) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.
(d) Calculate directly from Coulomb's law the exact potential for (b) in the x-y plane. Plot it as a function of distance and compare with the result found in (c).
Divide out the asymptotic form in parts (c) and (d) in order to see the behavior at large distances more clearly.

4.2 A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0)$$

4.5 A localized charge density $\rho(x, y, z)$ is placed in an external electrostatic field described by a potential $\Phi^{(0)}(x, y, z)$. The external potential varies slowly in space over the region where the charge density is different from zero.
(a) From first principles calculate the total *force* acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is

$$\mathbf{F} = q\mathbf{E}^{(0)}(0) + \{\nabla[\mathbf{p}\cdot\mathbf{E}^{(0)}(\mathbf{x})]\}_0 + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})\right]\right\}_0 + \cdots$$

Compare this to the expansion (4.24) of the *energy* W. Note that (4.24) is a number—it is not a function of $\mathbf{x}$ that can be differentiated! What is its connection to $\mathbf{F}$?

4.6 A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.
(a) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)_0$$ ← see p. 143 ed 2

(b) If it is known that $Q = 2\times10^{-24}$ cm² and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of e/a_0^3, where $a_0 = \hbar^2/me^2 = 0.529\times10^{-8}$ cm is the Bohr radius in hydrogen.
(c) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153} ($Z=63$) has a quadrupole moment $Q = 2.5\times10^{-24}$ cm² and a mean radius

$$R = (a+b)/2 = 7\times10^{-13}\text{ cm}$$

determine the fractional difference in radius $(a-b)/R$.

6.11 With the same assumptions as in Problem 6.10 discuss the conservation of angular momentum. Show that the differential and integral forms of the conservation law are

$$\frac{\partial}{\partial t}(\mathscr{L}_{\text{mech}}+\mathscr{L}_{\text{field}})+\nabla\cdot\overleftrightarrow{\mathbf{M}}=0$$

and

$$\frac{d}{dt}\int_V(\mathscr{L}_{\text{mech}}+\mathscr{L}_{\text{field}})\,d^3x+\int_S \mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}\,da=0$$

where the field angular-momentum density is

$$\mathscr{L}_{\text{field}}=\mathbf{x}\times\mathbf{g}=\frac{\mu\epsilon}{4\pi c}\mathbf{x}\times(\mathbf{E}\times\mathbf{H})$$

and the flux of angular momentum is described by the tensor

$$\overleftrightarrow{\mathbf{M}}=\overleftrightarrow{\mathbf{T}}\times\mathbf{x}$$

Note: Here we have used the dyadic notation for M_{ij} and T_{ij}. The double-headed arrow conveys a fairly obvious meaning. For example, $\mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}$ is a vector whose jth component is $\sum_i n_iM_{ij}$. The second-rank $\overleftrightarrow{\mathbf{M}}$ can be written as a third-rank tensor, $M_{ijk}=T_{ij}x_k-T_{ik}x_j$. But in the indices j and k it is antisymmetric and so has only three independent elements. Including the index i, M_{ijk} therefore has nine components and can be written as a pseudo tensor of the second rank, as above.

6.12 A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen.

(*a*) From the law of conservation of linear momentum show that the pressure (called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

(*b*) In the neighborhood of the earth the flux of electromagnetic energy from the sun is approximately 0.14 watt/cm^2. If an interplanetary "sailplane" had a sail of mass 10^{-4} gm/cm^2 of area and negligible other weight, what would be its maximum acceleration in centimeters per second squared due to the solar radiation pressure? How does this compare with the acceleration due to the solar "wind" (corpuscular radiation)?

6.19 An example of the preservation of causality and finite speed of propagation in spite of the use of the Coulomb gauge is afforded by a dipole source that is flashed on and off at $t=0$. The effective charge and current densities are

$$\rho(\mathbf{x},t)=\delta(x)\delta(y)\delta'(z)\delta(t)$$
$$J_z(\mathbf{x},t)=-\delta(x)\delta(y)\delta(z)\delta'(t)$$

where a prime means differentiation with respect to the argument. This dipole is of unit strength and it points in the negative z-direction.

(*a*) Show that the instantaneous Coulomb potential (6.45) is

$$\Phi(\mathbf{x},t)=-\delta(t)\frac{z}{r^3}$$

(*b*) Show that the *transverse* current $\mathbf{J}_t$ is

$$\mathbf{J}_t(\mathbf{x},t)=-\delta'(t)\left[\tfrac{2}{3}\boldsymbol{\epsilon}_3\delta(\mathbf{x})-\frac{\boldsymbol{\epsilon}_3}{4\pi r^3}+\frac{3}{4\pi r^3}\mathbf{n}(\boldsymbol{\epsilon}_3\cdot\mathbf{n})\right]$$

where the factor of 2/3 multiplying the delta function comes from treating the gradient of z/r^3 according to Eq. (4.20).

(*c*) Show that the electric and magnetic fields are causal and that the electric field components are

$$E_x(\mathbf{x},t)=\frac{c}{r}\left[-\delta''(r-ct)+\frac{3}{r}\delta'(r-ct)-\frac{3}{r^2}\delta(r-ct)\right]\sin\theta\cos\theta\cos\phi$$

E_y is the same as E_x, with $\cos\phi$ replaced by $\sin\phi$, and

$$E_z(\mathbf{x},t)=\frac{c}{r}\left[\sin^2\theta\,\delta''(r-ct)+(3\cos^2\theta-1)\cdot\left(\frac{\delta'(r-ct)}{r}-\frac{\delta(r-ct)}{r^2}\right)\right]$$

Hint: While the answer in (*b*) displays the transverse current explicitly, the less explicit form,

$$\mathbf{J}_t(\mathbf{x},t)=-\delta'(t)\left[\boldsymbol{\epsilon}_3\delta(\mathbf{x})+\frac{1}{4\pi}\nabla\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]$$

can be used with Eq. (6.69) to calculate the vector potential and the fields for part (*c*). An alternative method is to use the Fourier transforms in time of $\mathbf{J}_t$ and $\mathbf{A}$, the Green function (6.62) and its spherical wave expansion from Chapter 16.